# Formula Collection

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# Integrals and Identities

## Some integrals

### Indefinite Integrals

$$\int \frac{x}{ax+b}dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b|$$

$$\int \frac{1}{x(ax+b)}dx = -\frac{1}{b}\ln\left|\frac{ax+b}{x}\right|$$

$$\int \frac{ax+b}{fx+g}dx = \frac{ax}{f} + \frac{bf-ag}{f^2}\ln|fx+g|$$

$$\int \frac{x}{(ax+b)(fx+g)}dx = \frac{1}{bf-ag}\left[\frac{b}{a}\ln|ax+b| - \frac{g}{f}\ln|fx+g|\right]$$

$$\text{Definition:}\quad \chi = \begin{cases} \frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}} & \text{if } 4ac > b^2 \\ \frac{1}{a(p-q)}\ln\left|\frac{x-p}{x-q}\right| & \text{if } 4ac < b^2 \end{cases}$$

Where $p$ and $q$ are the roots of $ax^2+bx+c=0$.

$$\int \frac{1}{ax^2+bx+c}dx = \chi$$

$$\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln\left|ax^2+bx+c\right| - \frac{b}{2a}\chi$$

$$\int \frac{x^2}{ax^2+bx+c}dx = \frac{x}{a} - \frac{b}{2a^2}\ln\left|ax^2+bx+c\right| + \frac{b^2-2ac}{2a^2}\chi$$

$$\int \frac{1}{(ax^2+bx+c)^2}dx = \frac{2ax+b}{(4ac-b^2)(ax^2+bx+c)} + \frac{2a}{(4ac-b^2)}\chi$$

$$\int \frac{x}{(ax^2+bx+c)^2}dx = -\frac{bx+2c}{(4ac-b^2)(ax^2+bx+c)} - \frac{b}{(4ac-b^2)}\chi$$

$$\int \sqrt{ax+b}dx = \frac{2}{3a}(ax+b)^{3/2}$$

$$\int x\sqrt{ax+b}dx = \frac{2(3ax-2b)}{15a^2}(ax+b)^{3/2}$$

$$\int \frac{1}{\sqrt{ax+b}}dx = \frac{2\sqrt{ax+b}}{a}$$

$$\int \frac{x}{\sqrt{ax+b}}dx = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b}$$

$$\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}$$

$$\int x\sqrt{a^2-x^2}dx = -\frac{1}{3}(a^2-x^2)^{3/2}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a}$$

$$\int \frac{x}{\sqrt{a^2-x^2}}dx = -\sqrt{a^2-x^2}$$

$$\int \frac{1}{x\sqrt{a^2-x^2}}dx = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

$$\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left|x+\sqrt{x^2-a^2}\right|$$

$$\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left|x+\sqrt{x^2+a^2}\right|$$

$$\int \frac{1}{\sin ax}dx = \frac{1}{a}\ln\left|\tan\frac{ax}{2}\right|$$

$$\int \frac{1}{\cos ax}dx = \frac{1}{a}\ln\left|\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)\right|$$

$$\int \tan ax\, dx = -\frac{1}{a} \ln|\cos ax|$$

$$\int \cot ax\, dx = \frac{1}{a} \ln|\sin ax|$$

$$\int x \sin x\, dx = \sin x - x \cos x$$

$$\int x \sin^2 x\, dx = \frac{x^2}{4} - \frac{x \sin 2x}{4} - \frac{\cos 2x}{8}$$

$$\int \sin^2 x\, dx = \frac{1}{2}(x - \sin x \cos x)$$

$$\int \ln x\, dx = x \ln|x| - x$$

$$\int x e^{ax}\, dx = \frac{e^{ax}}{a^2}(ax - 1)$$

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)$$

**Definite Integrals**

$$\int_0^\infty x^{2n} \cdot e^{-ax^2}\, dx = \frac{(2n-1)!!}{2(2a)^n} \sqrt{\frac{\pi}{a}}$$

Where $a > 0$, $n! =$ negative integer.

$$\int_0^\infty x^{2n+1} \cdot e^{-ax^2}\, dx = \frac{n!}{2a^{n+1}}$$

$$\int_0^\infty x^k \cdot e^{-ax}\, dx = \Gamma(k+1) \cdot a^{-(k+1)}$$

$$I_k = \int_0^\infty \frac{x^k}{e^x - 1}\, dx = \Gamma(k+1) \cdot \zeta(k+1)$$

$$J_k = \int_0^\infty \frac{x^k}{e^x + 1}\, dx = (1 - 2^{-k}) \cdot I_k$$

$$\zeta(k) = \sum_{n=1}^\infty \frac{1}{n^k}$$

$$\int_0^{\pi/2} \sin^{2a+1} x \cdot \cos^{2b+1} x\, dx = \frac{\Gamma(a+1) \cdot \Gamma(b+1)}{2\Gamma(a+b+2)}$$

**Stirling's approximation**

$$N! \approx \sqrt{2\pi N}\left(\frac{N}{e}\right)^N \qquad N \gg 1$$

**Error Function**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}\, d\xi$$

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\xi^2}\, d\xi$$

$$\operatorname{erf}(\infty) = 1$$

## Power Series

**Power Series**

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...$$

$$\sin(x) = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - ...$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - ...$$

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{1}{15!}x^5 + ...|x| < \frac{\pi}{2}$$

$$\ln(1+x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + ...|x| < 1$$

$$(1+x)^a = 1 + \frac{a}{1!}x + \frac{a(a-1)}{2!}x^2 + ...$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + ...$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + ...$$

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - ...|x| < 1$$

$$\arcsin(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + ...|x| < 1$$

$$\cosh(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + ...$$

$$\sinh(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + ...$$

# Trigonometric Functions

## Trigonometric Functions

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

$$\sin(3\alpha) = 3\sin\alpha - 4\sin^3\alpha$$

$$\cos(3\alpha) = 4\cos^3\alpha - 3\cos\alpha$$

$$\sin^2\frac{\alpha}{2} = \frac{1}{2}(1 - \cos\alpha)$$

$$\cos^2\frac{\alpha}{2} = \frac{1}{2}(1 + \cos\alpha)$$

$$\sin\alpha + \cos\beta = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$$

$$\sin\alpha - \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$$

$$\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$$

$$\cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$$

$$\sin\alpha = \frac{1}{2i}(e^{i\alpha} - e^{-i\alpha})$$

$$\cos\alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha})$$

# Hyperbolic Functions

### Hyperbolic Functions

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

# Vector Analysis

## Vector Analysis

### Vector Products

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

where $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ are unit vectors.

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = 0$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c})$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = \boldsymbol{c}((\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{d}) - \boldsymbol{d}((\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c})$$

### Gradient, Divergence, Curl and The Laplace Operator

$$\mathrm{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\right)$$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$$\begin{aligned} \mathrm{div}\boldsymbol{a} = \nabla \cdot \boldsymbol{a} &= \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 a_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta a_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial a_\varphi}{\partial \varphi} \end{aligned}$$

$$
\begin{aligned}
\mathrm{rot}\boldsymbol{a} = \nabla \times \boldsymbol{a} &= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}, \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}, \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) \\
&= \left(\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta a_\varphi) - \frac{\partial a_\theta}{\partial\varphi}\right), \frac{1}{r\sin\theta}\frac{\partial a_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(ra_\varphi), \frac{1}{r}\frac{\partial}{\partial r}(ra_\theta) - \frac{1}{r}\frac{\partial a_r}{\partial\theta}\right)
\end{aligned}
$$

$$
\begin{aligned}
\Delta &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}
\end{aligned}
$$

$$
\Delta f(r) = \frac{1}{r}\frac{d^2}{dr^2}(rf), \quad r \neq 0
$$

$$
\begin{gathered}
\nabla \times (\nabla\mathcal{U}) = 0 \\
\nabla \cdot (\nabla\mathcal{U}) = \nabla\mathcal{U} \\
\nabla \cdot (\nabla \times \boldsymbol{A}) = 0 \\
\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \Delta\boldsymbol{A} \\
\nabla \cdot (\mathcal{U}V) = \mathcal{U}\Delta V + 2\nabla\mathcal{U} \cdot \nabla V + V\Delta\mathcal{U} \\
\nabla \cdot (\mathcal{U}\boldsymbol{V}) = \mathcal{U}\Delta\boldsymbol{V} + 2(\nabla\mathcal{U} \cdot \nabla)\boldsymbol{V} + \boldsymbol{V}\Delta\mathcal{U} \\
\nabla \times (\mathcal{U}\boldsymbol{V}) = \mathcal{U}\nabla \times \boldsymbol{V} + (\nabla\mathcal{U}) \times \boldsymbol{V} \\
\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B}) \\
\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} \\
\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A})
\end{gathered}
$$

**Gauss' theorem**

$$
\oint_{S(V)} \boldsymbol{a} \cdot d\boldsymbol{S} = \int_V (\nabla \cdot \boldsymbol{a}) dV
$$

Where $dV$ in polar coordinates are $r^2 \sin\theta \; dr \; d\theta \; d\varphi$

**Stoke's theorem**

$$
\oint_{C(S)} \boldsymbol{a} \cdot d\boldsymbol{l} = \int_S (\nabla \times \boldsymbol{a}) \cdot d\boldsymbol{S}
$$

Where $S$ is an arbitrary surface with border $C(S)$

**Green's theorem**

$$
\oint_{S(V)} (\Psi\nabla\varphi - \varphi\nabla\Psi) \cdot d\boldsymbol{S} = \int_V (\Psi\Delta\varphi - \varphi\Delta\Psi) dV
$$

# Electromagnetic field theory

## Statics

### Coulombs law

The force $F$ on a point charge $q_1$ at the point $\mathbf{r_1}$ caused by a point charge $q_2$ at the point $\mathbf{r_2}$

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 |r_1 - r_2|^2} \frac{(r_1 - r_2)}{|r_1 - r_2|}$$

### Electric Field Strength

From a point charge $q$ in $\boldsymbol{r'}$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0 R^2}\boldsymbol{e}_R$$

From charge distribution

$$\boldsymbol{E}(\boldsymbol{r}) = \int \frac{1}{4\pi\epsilon_0 R^2}\boldsymbol{e}_R dq(\boldsymbol{r'}),$$

$$dq(\boldsymbol{r'}) = \begin{cases} \rho_{tot}(\boldsymbol{r'})dv' = \rho(\boldsymbol{r'}) + \rho_p(\boldsymbol{r'}))dv' \\ \rho_{tot,s}(\boldsymbol{r'})dS' = \rho_s(\boldsymbol{r'}) + \rho_{p,s}(\boldsymbol{r'}))dS' \\ \rho_l(\boldsymbol{r'})dl' \end{cases}$$

From point dipole $\boldsymbol{p} = p\boldsymbol{e}_z$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos(\theta)\boldsymbol{e}_r + \sin(\theta)\boldsymbol{e}_\theta)$$

From line charge $\rho_l$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{\rho_l}{2\pi\epsilon_0 r_c}\boldsymbol{e}_{r_c}$$

From dipole line $\boldsymbol{p}_l = p_l\boldsymbol{e}_x$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{p_l}{2\pi\epsilon_0 r_c^2}(\cos(\varphi)\boldsymbol{e}_{r_c} + \sin(\varphi)\boldsymbol{e}_\varphi)$$

### Electrical Potential

$$\boldsymbol{E} = -\boldsymbol{\nabla} V$$

From pointsource $q$ in $\boldsymbol{r'}$

$$V(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0 R}$$

From charge distribution

$$V(\boldsymbol{r}) = \int \frac{1}{4\pi\epsilon_0 R} dq(\boldsymbol{r}')$$

From point dipole $\boldsymbol{p} = p\boldsymbol{e}_z$

$$V(\boldsymbol{r}) = \frac{\boldsymbol{p} \cdot \boldsymbol{r}}{4\pi\epsilon_0 r^3} = \frac{p\cos(\theta)}{4\pi\epsilon_0 r^2}$$

From line charge $\rho_l$

$$V(\boldsymbol{r}) = \frac{\rho_l}{2\pi\epsilon_0} \ln\left(\frac{1}{r_c}\right)$$

From dipole line $\boldsymbol{p}_l = p_l\boldsymbol{e}_x$

$$V(\boldsymbol{r}) = \frac{p_l}{2\pi\epsilon_0} \frac{\cos(\varphi)}{r_c}$$

### Electrical flow density

Where $\boldsymbol{D}$ is defined by $\boldsymbol{\nabla D} = \rho$

Gauss law, where $\boldsymbol{e}_n$ is the unit normal to the volume surface pointing outwards $\boldsymbol{P}, \boldsymbol{E}$ and $\boldsymbol{D}$:

$$\begin{cases} \boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} & \text{(valid generally)} \\ \boldsymbol{D} = \epsilon_r \epsilon_0 \boldsymbol{E} \end{cases}$$

### Polarization Charge

$$\rho_p = -\boldsymbol{\nabla} \cdot \boldsymbol{P} \qquad \text{space charge density}$$

$$\rho_{p,s} = \boldsymbol{e}_{n1} \cdot (\boldsymbol{P}_1 - \boldsymbol{P}_2) \qquad \text{surface charge density}$$

where the unit normal $\boldsymbol{e}_{n1}$ is directed from 1 to 2.

### Boundary Conditions

$$\begin{cases} E_t \text{ continuous} \\ \rho_s = \boldsymbol{e}_{n2} \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) \end{cases}$$

where $\rho_s$ is free surface charge density and $\boldsymbol{e}_{n2}$ is directed from volume 2 to volume 1.

**Electrostatic Energy**

$$W_e = \frac{1}{2}\sum_i Q_i V_i$$

$$W_e = \frac{1}{2}\int \rho V \, dv$$

$$W_e = \frac{1}{2}\int \boldsymbol{E}\cdot\boldsymbol{D}\, dv$$

**Maxwell's voltage**

$$|\boldsymbol{T}| = \frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D} \qquad \boldsymbol{E} \text{ is a bisector to } \boldsymbol{e}_n \text{ and } \boldsymbol{T}$$

**Torque on Electrical Dipole**

$$\boldsymbol{T}_e = \boldsymbol{p}\times\boldsymbol{E}$$

## DC Current

**Current Density**

$$I = \int \boldsymbol{J}\cdot e_n \, dS$$

**Conservation Equation**

$$\boldsymbol{\Delta}\cdot\boldsymbol{J} + \frac{\partial\rho}{\partial t} = 0$$

$$\oint \boldsymbol{J}\cdot\boldsymbol{e}_n \, dS = -\frac{dQ}{dt}$$

**Conductivity**

$$\boldsymbol{J} = \sigma\boldsymbol{E}$$

**Effect**

$$P = \int \boldsymbol{J}\cdot\boldsymbol{E}\, dv$$

**Boundary Conditions**

$$\begin{cases} \boldsymbol{e}_{n2}\cdot(\boldsymbol{J}_1 - \boldsymbol{J}_2) = 0 \quad \text{(no surface current)} \\ \boldsymbol{E}_{t1} = \boldsymbol{E}_{t2} \end{cases}$$

**Time Constant**

$$RC = \frac{\epsilon_r \epsilon_0}{\sigma}$$

**Analogy Elektrostatics - DC Current**

| | |
|---|---|
| $\boldsymbol{E}, V$ | $\boldsymbol{E}, V$ |
| $\boldsymbol{D}$ | $\boldsymbol{J}$ |
| $\epsilon_r \epsilon_0$ | $\sigma$ |
| $Q$ | $I$ |
| $C$ | $G$ |

## Magnetostatics

### Magnetic Flow Density

From point dipole $\boldsymbol{m} = m\boldsymbol{e}_z$:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta \boldsymbol{e}_r + \sin\theta \boldsymbol{e}_\theta)$$

From current density $\boldsymbol{J}_{tot}(\boldsymbol{r'})$:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}_{tot}(\boldsymbol{r'}) \times \boldsymbol{e}_R}{R^2}\, dv'$$

where $\boldsymbol{J}_{tot} = \boldsymbol{J} + \boldsymbol{J}_m$. From current line:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{I\, d\boldsymbol{l'} \times \boldsymbol{e}_R}{R^2}$$

From circular thread loop:

$$\boldsymbol{B}(x=0, y=0, z) = \frac{\mu_0 I}{2} \frac{b^2}{(b^2+z^2)^{3/2}} \boldsymbol{e}_z$$

From coil:

$$\boldsymbol{B} = \frac{\mu_0 N I}{\ell} \frac{\cos(\alpha_2) - \cos(\alpha_1)}{2} \boldsymbol{e}_z$$

From long straight current path:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{2\pi r_c} \boldsymbol{e}_\varphi$$

## Vector Potential

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

From current density $\boldsymbol{J}_{tot}(\boldsymbol{r'})$:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}_{tot}(\boldsymbol{r'})}{R}\, dv'$$

From current line:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{I\, d\boldsymbol{l'}}{R}$$

From long straight current path:

$$\boldsymbol{A} = \frac{\mu_0 I}{2\pi} \ln\left(\frac{1}{r}\right) \boldsymbol{e}_z$$

From point dipole :

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3}$$

### Magnetic Flow

$$\Phi = \int \boldsymbol{B} \cdot \boldsymbol{e}_n\, dS = \oint \boldsymbol{A} \cdot dl$$

### Linked Flow

$$\Lambda = N\Phi$$

### Self-Inductance and Mutual Inductance

$$\Lambda_1 = L_1 I_1 + M I_2$$
$$\Lambda_2 = L_2 I_2 + M I_1$$

### Magnetic Field Strength

Amperes Law:

$$\oint \boldsymbol{H} \cdot d\ell = \int \boldsymbol{J} \cdot \boldsymbol{e}_n\, dS = I_{\text{inside}}$$

Connection between magnetization $\boldsymbol{M}, \boldsymbol{B}$ and $\boldsymbol{H}$:

$$\begin{cases} \boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) & \text{(holds generally)} \\ \boldsymbol{B} = \mu_r \mu_0 \boldsymbol{H} \end{cases}$$

**Equivalent Current Density**

$$\boldsymbol{J}_m = \boldsymbol{\nabla} \times \boldsymbol{M} \qquad \text{volume current density}$$

$$\boldsymbol{J}_m = \boldsymbol{\nabla} \times \boldsymbol{M} \qquad \text{surface current density}$$

**Boundary Conditions**

$$\begin{cases} \boldsymbol{e}_{n2} \times (\boldsymbol{H}_1 - H_2) = \boldsymbol{J}_s \\ \boldsymbol{B}_n \text{ Continuous} \end{cases}$$

**Scalar Potential**

From magnetic dipole $\boldsymbol{m}$:

$$V_m = \frac{1}{4\pi} \frac{\boldsymbol{m} \cdot \boldsymbol{e}_R}{R^2}$$

**Magnetic Pole Density**

$$\begin{cases} \rho_m = -\boldsymbol{\nabla} \cdot \boldsymbol{M} & \text{volume pole density} \\ \rho_{m,s} = e_{n1} \cdot (\boldsymbol{M}_1 - \boldsymbol{M}_2) & \text{surface pole density} \end{cases}$$

**Magnetic Force Law**

$$d\boldsymbol{F}_m = I d\boldsymbol{l} \times \boldsymbol{B}$$

**Magetic moment for current loop**

$$\boldsymbol{m} = \int I \boldsymbol{e}_n \, dS$$

**Torque on Magnetic Moment**

$$\boldsymbol{T}_m = \boldsymbol{m} \times \boldsymbol{B}$$

**Maxwell's Voltage**

$$|\boldsymbol{T}| = \frac{1}{2} \boldsymbol{B} \cdot \boldsymbol{H} \qquad \boldsymbol{B} \text{ is a bisector to } \boldsymbol{e}_n \text{ and } \boldsymbol{T}$$

**Magnetic Energy**

$$W_m = \frac{1}{2} \int \boldsymbol{J} \cdot \boldsymbol{A} \, dv = \frac{1}{2} \int \boldsymbol{B} \cdot \boldsymbol{H} \, dv = \frac{1}{2} \sum_i \sum_j L_{ij} I_i I_j$$

Two coils:

$$W_m = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2$$

**Reluctance**

$$R = \frac{1}{\mu_r \mu_0 S}$$

## Electromagnetic Fields

## Induced emk

$$\mathcal{E} = \oint (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot d\ell$$

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

$$\mathcal{E} = -\frac{d\Lambda}{dt} \quad \text{(coil with multiple turns)}$$

**Maxwell's equations**

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\boldsymbol{\nabla} \cdot D = \rho$$

$$\boldsymbol{\nabla} \boldsymbol{B} = 0$$

**The Conservation Equation**

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$$

**Potentials**

$$V(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho\left(\boldsymbol{r'}, t - \frac{R}{c}\right)}{R} \, dv' = \frac{1}{4\pi\epsilon_0} \int \frac{\rho_{ret}}{R} \, dv'$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}\left(\boldsymbol{r'}, t - \frac{R}{c}\right)}{R} \, dv' = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}_{ret}}{R} \, dv'$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

$$\boldsymbol{E} = -\boldsymbol{\nabla} V - \frac{\partial \boldsymbol{A}}{\partial t}$$

**Magnetic Flow Density**

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}_{ret}\times \boldsymbol{e}_R}{R^2}\,dv' + \frac{\mu_0}{4\pi c}\int \frac{\boldsymbol{J}'_{ret}\times \boldsymbol{e}_R}{R}\,dv'$$

**Filamentuos Antenna**

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\int \frac{i(z,t-R/c)d\boldsymbol{l}\times \boldsymbol{e}_R}{R^2} + \frac{\mu_0}{4\pi c}\int \frac{i(z,t-R/c)d\boldsymbol{l}\times \boldsymbol{e}_R}{R}$$

**Oscillating Electric Dipole**

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{p}'(t-R/c)\times \boldsymbol{e}_R}{R^2} + \frac{\mu_0}{4\pi c}\frac{\boldsymbol{p}''(t-R/c)\times \boldsymbol{e}_R}{R}$$

**Oscillating Magnetic Dipole**

$$\boldsymbol{B} = -\frac{\mu_0}{4\pi}\frac{\boldsymbol{m}'(t-R/c)\times \boldsymbol{e}_R}{R^2} - \frac{\mu_0}{4\pi c}\frac{\boldsymbol{m}''(t-R/c)\times \boldsymbol{e}_R}{R}$$

**Pointing's Vector**

$$\boldsymbol{P}_S(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t)\times \boldsymbol{H}(\boldsymbol{r},t)$$

## Time Harmonic Fields

**Planar Sinusoidal Wave**

$$\boldsymbol{E} = \hat{E}\cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r} + \phi)\boldsymbol{e}_E \quad \text{instantanious value}$$

$$\boldsymbol{E} = E_0 e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{e}_E \quad \text{complex value}$$

$$E_0 = \hat{E}e^{j\phi} \quad \text{top value scale}$$

$$E_0 = \frac{\hat{E}}{\sqrt{2}}e^{j\phi} \quad \text{effective value scale}$$

**Propagation Rate**

$$v = \frac{1}{\sqrt{\mu_0\mu_r\epsilon_0\epsilon_r}} \quad v = \frac{\omega}{k} \quad k = |\boldsymbol{k}|$$

**Wave Impedance Non-Conductive Space**

$$\eta = \sqrt{\frac{\mu_r\mu_0}{\epsilon_r\epsilon_0}}$$

**Rule of Right-Hand Systems**

$$\boldsymbol{e}_k = \boldsymbol{e}_E \times \boldsymbol{e}_H \quad E = \eta H \qquad \boldsymbol{e}_k = \boldsymbol{e}_E \times \boldsymbol{e}_B \quad E = vB$$

**Planar Wave in Space with Condctivity**

$$\boldsymbol{E} = E_0 e^{\gamma z} \boldsymbol{e}_x$$

**Complex Propagation Constant**

$$\gamma = \sqrt{j\omega\mu_r\mu_0(\sigma + j\omega\epsilon_r\epsilon_0)} \qquad \gamma = \alpha j\beta$$

**Waveinpedance, Space With Given Conductivity**

$$\eta = \sqrt{\frac{j\omega\mu_r\mu_0}{\sigma + j\omega\epsilon_r\epsilon_0}}$$

**Penetration Depth**

$$\delta = \sqrt{\frac{2}{\omega\mu_r\mu_0\sigma}}$$

## Derivatives

**Derivatives**

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) = \frac{1}{1+x^2}$$

# Fourier Analysis

## Fourier Analysis Introduction

**Fourier Sum**

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{m=1}^{\infty} b_m \sin(m\omega t)$$

**Fourier Coefficients**

$$a_0 = \frac{2}{T}\int_0^T f(t)dt$$

$$a_m = \frac{2}{T}\int_0^T f(t)\cos(m\omega t)dt$$

$$b_m = \frac{2}{T}\int_0^T f(t)\sin(m\omega t)dt$$

**Rewriting with Eulers Formula**

$$f(t) = \sum_{m=-\infty}^{m=\infty} c_n e^{-im\omega t}$$

$$c_m = \frac{1}{T}\int_0^T f(t)e^{im\omega t}dt$$

**For Non-periodic Functions**

$$f(t) = \int_{-\infty}^{\infty} g(\omega)e^{-i\omega t}d\omega$$

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$$

**Fourier s integral theorem**

$$f(\boldsymbol{r}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} A(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r}}d^3k$$

$$A(\boldsymbol{k}) = \int_{-\infty}^{\infty} f(\boldsymbol{r})e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, d^3r$$

**Periodic boundaries**

If the function $f(\boldsymbol{r})$ is such that

$$f(\boldsymbol{r}) = f(\boldsymbol{r} + L\boldsymbol{R})$$

[For some positive integer $L$ and lattice vector $\boldsymbol{R}$. Then/Får något positivt heltal $L$ och gittervektor $\boldsymbol{R}$. Då håller att]]

$$f(\boldsymbol{r}) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{k}=\frac{\boldsymbol{G}}{L}} c_{\boldsymbol{k}} \cdot e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$c_{\boldsymbol{k}} = \frac{1}{\sqrt{V}}\int_V f(\boldsymbol{r})\cdot e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, d^3r$$

Where $\boldsymbol{G}$ is the reciprocal lattice vector and $V = L^3|\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)| = L^3 V_a$. The functions $\frac{1}{\sqrt{V}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ is a complete orthonormal basis in $V$.. If the volume $V$ is large, the sum can be replaced by an integral:

$$\sum_{\boldsymbol{k}} \rightarrow \frac{V}{(2\pi)^3} \int d^3k$$

**Dirac Delta Function**

$$\int_A^B f(x)\delta(x - x_0)\, dx = \begin{cases} f(x_0) & \text{if } A < x_0 < B \\ 0 & \text{otherwise} \end{cases}$$

If $f(x)$ is a "nice" function.

$$\delta(f(x)) = \sum_{\forall i; f(x_i)=0, f'(x_i)\neq 0} \frac{1}{|f'(x_i)|} \delta(x - x_i)$$

$$\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x_0)}\, dk$$

**Kronecker Delta**

$$\delta_{n,m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\phi(n-m)}\, d\phi = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

# Mechanics

## Mechanics

### Momentary Speed

$$v = \frac{dx}{dt}$$

### Momentary Acceleration

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

### Momentum

$$\mathbf{p} = \mathbf{m} \cdot \mathbf{v}$$

### Force

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m \cdot \mathbf{a}$$

**Gravitation**

$$F = C \cdot \frac{m_1 \cdot m_2}{r^2}$$

**Centripetal Acceleration**

$$a_c = \frac{v^2}{r} = r\omega^2$$

**Work**

$$W = \int_{x_1}^{x_2} F(x)\, dx$$

**Kinetic Energy**

$$K = \frac{m \cdot v^2}{2}$$

**Potential Energy**

$$W = -\Delta U,\ F = -\frac{dU}{dx}$$

**Reduced Mass**

$$\frac{1}{\mu} = \frac{1}{m} + \frac{1}{M}$$

**Angular Frequency**

$$\omega = 2\pi f = \frac{2\pi}{T}$$

# Quantum Mechanics

## Quantum Mechanics

### Schrödinger Equation

$$H\boldsymbol{\psi}(\boldsymbol{r}, t) = \left[-\frac{\hbar^2}{2m}\Delta + \mathcal{U}(\boldsymbol{r})\right]\boldsymbol{\psi}(\boldsymbol{r}, t) = i\hbar\frac{\partial}{\partial t}\boldsymbol{\psi}(\boldsymbol{r}, t)$$

Where $H$ is a hamiltonian operator. If $H$ is time independent separation of variables gives:

$$\boldsymbol{\psi}(\boldsymbol{r}, t) = \boldsymbol{\Phi}(\boldsymbol{r}) \cdot e^{-\frac{i}{\hbar}Et}$$
$$\left[-\frac{\hbar^2}{2m}\Delta + \mathcal{U}(\boldsymbol{r})\right]\boldsymbol{\Phi}(\boldsymbol{r}) = E\boldsymbol{\Phi}(\boldsymbol{r})$$

The general time dependent solution is:

$$\boldsymbol{\psi}(\boldsymbol{r},t) = \sum_n a_n \cdot \boldsymbol{\Phi}(\boldsymbol{r}) e^{-\frac{i}{\hbar}Et}$$

Where $a_n$ are found through the boundary conditions ($t = 0$):

$$a_n = \int \boldsymbol{\Phi}_n * (\boldsymbol{r}) \cdot \boldsymbol{\psi}(\boldsymbol{r}, t=0)\, d^3r$$

## Operators

### Linear Operator

$$F(a\boldsymbol{\Phi}_1 + b\boldsymbol{\Phi}_2) = a \cdot F\boldsymbol{\Phi}_1 + b \cdot F\boldsymbol{\Phi}_2 \quad \forall \boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2$$

### Eigenvalue, Eigenfunction

$$Fu_n = f_n u_n$$

$u_n$ is a eigen function to the operator $F$ with corresponding eigenvalue $f_n$.

### Hermitian Operator

$$\int (Hu) * v\, d^3r = \int u * Hv\, d^3r, \quad \forall u, v$$

A hermitian operator has real eigenvalues and corresponding eigenfunctions can be choosen to be orthonormal. Practically all operators in quantum mechanics are linear and hermitian.

### Eigenfunction Expansion

$$\boldsymbol{\psi}(\boldsymbol{r}) = \sum_n a_n m \cdot u_n(\boldsymbol{r}), \quad a_n = \int u_n * \cdot \boldsymbol{\psi} \cdot d^3r$$

### Expansion Postulate

At a meassurement of an observable $F$ on a system described by a wavefunction $\boldsymbol{\psi}$ only eigenvalues of the operator $F$ can be found. The probability of the result $F = f_n$ is given by

$$P(F = f_n) = \left| \int u_n * \boldsymbol{\psi}\, d^3r \right|^2, \quad Fu_n = f_n u_n$$

**Momentum Operators**

$$L^2 = -\hbar^2 \left[ \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial}{\partial\theta} \right) \right]$$

$$L_z = \frac{\hbar}{i} \frac{\partial}{\partial\varphi}$$

$L^2$ and $L_z$ have normalized eigenfunctions $\Upsilon_l^m(\theta,\varphi)$ for which it holds that:

$$L^2 \Upsilon_l^m = \hbar^2 l(l+1) \Upsilon_l^m$$

$$L_z \Upsilon_l^m = m\hbar \Upsilon_l^m$$

| $l$ | $m$ | $\Upsilon_l^m(\theta,\varphi)$ |
|---|---|---|
| 0 | 0 | $\Upsilon_0^0 = \frac{1}{\sqrt{4\pi}}$ |
| 1 | 0 | $\Upsilon_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta$ |
| 1 | $\pm1$ | $\Upsilon_1^{\pm1} = \pm\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\varphi}$ |
| 2 | 0 | $\Upsilon_2^0 = \sqrt{\frac{5}{16\pi}} \left(3\cos^2\theta - 1\right)$ |
| 2 | $\pm1$ | $\Upsilon_2^{\pm1} = \pm\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{\pm i\varphi}$ |
| 2 | $\pm2$ | $\Upsilon_2^{\pm2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{\pm 2i\varphi}$ |

**Commutators and Momentum Operators**

$$\epsilon_{ijk} = \begin{cases} 1 & ijk \text{ even} \\ -1 & ijk \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$[x_i, p_j] = i\hbar \cdot \delta_{ij}$$

$$[x_i, L_j] = i\hbar \cdot \epsilon_{ijk} \cdot x_k$$

$$[L_i, L_j] = i\hbar \cdot \epsilon_{ijk} \cdot L_k$$

$$[x_i, x_j] = [p_i, p_j] = 0$$

$$[p_i, L_j] = i\hbar \cdot \epsilon_{ijk} \cdot p_k$$

$$J_+ = J_x + iJ_y$$
$$J_- = J_x - iJ_y$$

$$J_\pm J_\mp = J^2 - J_z^2 \pm \hbar \cdot J_z$$

$$[J_+, J_-] = 2\hbar \cdot J_z$$

$$[J_z, J_\pm] = \pm\hbar \cdot J_\pm$$

$$J_+ \phi_{j,m} = \sqrt{(j-m)(j+m+1} \cdot \hbar \cdot \phi_{j,m+1}$$

$$J_- \phi_{j,m} = \sqrt{(j+m)(j-m+1} \cdot \hbar \cdot \phi_{j,m-1}$$

$$\Upsilon_l^l(\theta, \varphi) = (-1)^l \sqrt{\frac{2l+1)}{4\pi} \frac{(2l)!}{2^{2l}(l!)^2}} \cdot \sin^l \theta \cdot e^{il\varphi}$$

## Applications

### 0.0.1 Low potential with infinitely rigid walls in one dimension

$$\mathcal{U}(x) = \begin{cases} \infty & x \leq 0, \ a \leq x \\ 0 & 0 < x < a \end{cases}$$

$$\Phi_n(x) = \begin{cases} 0 & \text{for } x \leq 0 \text{ and } a \leq x \\ \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} & \text{for } 0 < x < a \end{cases}$$

$$E_n = \frac{\pi^2 \hbar^2 n^2}{2ma^2}$$

**Harmonic Oscillator 1D**

$$\mathcal{U}(x) = \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}kx^2$$

$$N_n = (2^n n!)^{-1/2} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

Hermites polynom:

$$H_n(\xi) = (-1)^n \cdot e^{\xi^2} \cdot \frac{d^n e^{-\xi^2}}{d\xi^n}$$

$$\mathbf{\Phi}_n(x) = N_n \cdot e^{-\frac{m\omega}{2\hbar}x^2} \cdot H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$$

$$E_n = \hbar\omega \cdot \left(n + \frac{1}{2}\right)$$

The wave equations can alternatively be written:

$$u_n(x) = N\left(\frac{\partial}{\partial x} - ax\right)^n \cdot u_0(x)$$

$$u_0(x) = e^{-ax^2/2}$$

**Spherical Symmetric Potential**

$$\mathcal{U}(\boldsymbol{r}) = \mathcal{U}(r)$$

$$H = -\frac{\hbar}{2mr^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\right] + \frac{L^2}{2mr^2} + \mathcal{U}(r)$$

$$H\boldsymbol{\psi}_{nlm}(\boldsymbol{r}) = E_{nlm}\boldsymbol{\psi}_{nlm}(\boldsymbol{r})$$

$$\boldsymbol{\psi}_{nlm}(\boldsymbol{r}) = \frac{G_{nl}(r)}{r}\Upsilon_l^m(\theta, \phi)$$

Radial equation:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}G(r) + \left[\frac{l(l+1)\hbar^2}{2mr^2} + \mathcal{U}(r)\right]G(r) = EG(r)$$

**Hydrogen-like Atom**

$$\mathcal{U}(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}$$

The Schrödinger equation simplifies to:

$$\left[\Delta + \frac{2Z}{a_0 r} + \frac{2mE}{\hbar^2}\right]\mathbf{\Phi}(r) = 0$$

Radial wave functions of hydrogenic atoms:

| $n$ | $l$ | $R_{nl}(r)$ |
|---|---|---|
| 1 | 0 | $R_{10}(r) = 2\left(\frac{Z}{a_0}\right)^{3/2} e^{-\rho/2}$ |
| 2 | 0 | $R_{20}(r) = \frac{1}{2\sqrt{2}}\left(\frac{Z}{a_0}\right)^{3/2}(2-\rho)e^{-\rho/2}$ |
| 2 | 1 | $R_{21}(r) = \frac{1}{2\sqrt{6}}\left(\frac{Z}{a_0}\right)^{3/2}\rho e^{-\rho/2}$ |
| 3 | 0 | $R_{30}(r) = \frac{1}{9\sqrt{3}}\left(\frac{Z}{a_0}\right)^{3/2}\left(6-6\rho+\rho^2\right)e^{-\rho/2}$ |
| 3 | 1 | $R_{31}(r) = \frac{1}{9\sqrt{6}}\left(\frac{Z}{a_0}\right)^{3/2}\rho(4-\rho)e^{-\rho/2}$ |
| 3 | 2 | $R_{32}(r) = \frac{1}{9\sqrt{30}}\left(\frac{Z}{a_0}\right)^{3/2}\rho^2 e^{-\rho/2}$ |

$$E-n = -\frac{mZ^2e^4}{32\pi^2\epsilon_0^2\hbar^2n^2} = -\frac{Z^2\hbar^2}{2a_0^2mn^2} = -13.6\frac{Z^2}{n^2}\mathrm{eV}$$

$$S(x,t) = \frac{\hbar}{2im}\left[\boldsymbol{\psi}^*\cdot\frac{\partial\boldsymbol{\psi}}{\partial x} - \boldsymbol{\psi}\frac{\partial}{\partial x}\boldsymbol{\psi}^*\right]$$

## Disturbance Calculations

Time independent disturbance:

$$\left.\begin{array}{l}\left(H^0+H'\right)\boldsymbol{\psi}'_m = E'_m\boldsymbol{\psi}'_m\\ H^0\boldsymbol{\psi}_n = E_n^0\boldsymbol{\psi}_n\end{array}\right\} \Longrightarrow$$

$$E'_m = E_m^0 + \langle m|H'|m\rangle + \sum_{n\neq m}\frac{|\langle m|H'|n\rangle|^2}{E_m^0 - E_n^0}$$

$$\boldsymbol{\psi}'_m = \boldsymbol{\psi}_m + \sum_{n\neq m}\frac{\int\boldsymbol{\psi}_n^* H'\boldsymbol{\psi}_m\, d^3r}{E_m^0 - E_n^0}\boldsymbol{\psi}_n$$

Time dependent disturbance:

$$\left.\begin{array}{l} H = H^0 + H' \\ H^0 \text{ Time independent} \\ H^0\boldsymbol{\psi}_n = E_n^0\boldsymbol{\psi}_n \\ H\boldsymbol{\psi}' = i\hbar\frac{\partial}{\partial t}\boldsymbol{\psi}' \end{array}\right\} \Longrightarrow$$

$$\boldsymbol{\psi}'_m = \sum_n a_{mn}(t)\boldsymbol{\psi}_n$$

$$\dot{a}_{mn} = -\frac{i}{\hbar}e^{-i(E_m-E_n)t/\hbar} \cdot H'_{nm}$$

**"Golden Rule"**

The transition probability per unit of time $w_{f\leftarrow i}$ for a transition from the state $\boldsymbol{\psi}_i$ to a group of states $F = \{\boldsymbol{\psi}_f\}$ with energy sin $E_i^0$ for a system characterized by the state density $\rho(E)$ is given by:

$$w_{f\leftarrow i} = \frac{2\pi}{\hbar}|\left\langle f|H'|i\right\rangle|^2_{E_i^0\approx E_f^0} \cdot \rho(E_f^0)$$

**Dispersion (Born Approximation)**

$$\frac{d\sigma}{d\Omega} = |f(\xi,\eta)|^2$$

$$f(\xi,\eta) = \frac{m}{2\pi\hbar^2}\int e^{i(\boldsymbol{k}_i-\boldsymbol{k}_f)\cdot\boldsymbol{r}} \cdot v(\boldsymbol{r})\, d^3r$$

For spherical symmetrical potential:

$$f(\xi,\eta) = \frac{2m}{\hbar^2 K}\int_0^\infty \sin(Kr)\cdot r\cdot v(r)dr, \qquad |K| = 2k\cdot\sin\left(\frac{\xi}{2}\right)$$

Spherical box-potential:

$$V(r) = \begin{cases} -V_0 & r \le a \\ 0 & r > a \end{cases}$$

$$f(\xi,\eta) = -\frac{2mV_0}{\hbar^2}\cdot\frac{\sin(Ka) - Ka\cos(Ka)}{K^3}$$

Screened Coulomb Potential:

$$v(r) = -\frac{A}{r}\cdot e^{-\alpha r}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mA}{\hbar^2\left(\alpha^2 + 4k^2\sin^2(\xi/2)\right)}\right)^2$$

$$\sigma = \left(\frac{Am}{\hbar^2}\right)^2 \frac{16\pi}{\alpha^2\left(\alpha^2+4k^2\right)}$$

$$\text{When } \alpha \to 0, \quad \frac{d\sigma}{d\Omega} \to \left(\frac{Am}{\hbar^2}\right)^2 \frac{1}{4\left(k\sin(\xi/2)\right)^4}$$

**Periodic Potential**

$$V(x) = \left.\begin{matrix} 0 & n(a+b) < x < n(a+b)+a \\ V_0 & n(a+b)+a < x < (n+1)(a+b) \end{matrix}\right\}$$

Continuity Requirements:

$$\cos k_1 a \cdot \cos k_2 b - \frac{k_1^2+k_2^2}{2k_1k_2}\sin k_1 a \cdot \sin k_2 b = \cos(k(a+b)), \quad V_0 < E$$

$$\cos k_1 a \cdot \cosh \kappa b - \frac{k_1^2+\kappa^2}{2k_1\kappa}\sin k_1 a \cdot \sinh \kappa b = \cos(k(a+b)), \quad V_0 < E$$

Phase and group speed:

$$v_f = \frac{\omega}{k}, \quad v_g = \frac{d\omega}{dk} = \frac{dE}{dp}$$

Effective mass:

$$m^* = \left(\frac{1}{\hbar^2}\frac{d^2E}{dk^2}\right)^{-1}$$

# Atomic Physics

## Atomic Physics

### DeBroglie wavelength

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

### Rydberg

$$\tilde{\nu} = \frac{1}{\lambda} = R\left(\frac{1}{n^2} - \frac{1}{m^2}\right)Z^2$$

$$E_n = -hcR\frac{Z^2}{n^2}$$

$$hcR_\infty = \frac{m_e(e^2/4\pi\epsilon_0)^2}{2\hbar^2} = 13.606 \text{ eV}$$

$$R = R_\infty \cdot \frac{M_N}{m_e + M_N}$$

**Alkaline-like System**

$$n^* = n - \delta_l$$

$$E = -hcR_\infty \frac{Z_0^2}{n^{*2}}$$

$$\Delta E_{FS} = -\frac{Z_i^2 Z_0^2}{n^{*3} l(l+1)} \alpha^2 hcR_\infty$$

**Hydrogen-like Atoms**

$$V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r} = -\frac{Ze_0^2}{r}$$

$$a_0 = \frac{\hbar^2 4\pi\epsilon_0}{m_e e^2}$$

In Bohrs model of the atom: $r_n = a_0 n^2/Z$

**Radial Functions for Hydrogen-like Systems**

$$R_{1,0} = \left(\frac{Z}{a_0}\right)^{3/2} 2e^{-Zr/a_0}$$

$$R_{2,0} = \left(\frac{Z}{2a_0}\right)^{3/2} 2\left(1 - \frac{Zr}{2a_0}\right) e^{-Zr/2a_0}$$

$$R_{2,0} = \left(\frac{Z}{2a_0}\right)^{3/2} \frac{2}{\sqrt{3}} \frac{Zr}{2a_0} e^{-Zr/2a_0}$$

**Spherical Surface Functions**

| $l$ | $m$ | $\Upsilon_l^m(\theta,\varphi)$ |
|---|---|---|
| 0 | 0 | $\Upsilon_0^0 = \frac{1}{\sqrt{4\pi}}$ |
| 1 | 0 | $\Upsilon_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$ |
| 1 | $\pm1$ | $\Upsilon_1^{\pm1} = \pm\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi}$ |
| 2 | 0 | $\Upsilon_2^0 = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$ |
| 2 | $\pm1$ | $\Upsilon_2^{\pm1} = \pm\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\varphi}$ |
| 2 | $\pm2$ | $\Upsilon_2^{\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\varphi}$ |

**Hamilton Operator for Multi-electron Systems**

$$\boldsymbol{H} = \sum_{i=1}^{N}\left(-\frac{\hbar^2}{2m}\nabla_i^2 - \frac{Ze^2/4\pi\epsilon_0}{r_i} + \sum_{j>i}^{N}\frac{e^2/4\pi\epsilon_0}{r_{ij}}\right)$$

$$\langle LM_L|l_1|LM_L\rangle = \frac{\langle l_1\cdot\boldsymbol{L}\rangle}{L(L+1)}\langle LM_L|\boldsymbol{L}|LM_L\rangle$$

**LS coupling**

$$\text{Terms:}\quad \begin{cases} L = |l_1 - l_2|, ..., l_1 + l_2 \\ S = |s_1 - s_2|, ..., s_1 + s_2 \end{cases}$$

$$\text{Levels:}\quad J = |L - S|, ..., L + S$$

**Zeeman Effect**

$$E_{ZE} = \begin{cases} g_J\mu_B BM_J & \text{(fine structure)} \\ g_F\mu_B BM_F & \text{(weak field, hfs)} \\ g_J\mu_B BM_J + AM_IM_J & \text{(strong field, } \mu_B B > A\text{)} \end{cases}$$

**Connection between magnetic moment and momentum**

$$g_S = 2$$

$$g_J = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)}$$

$$g_F = g_J \frac{F(F+1) + J(J+1) - I(I+1)}{2F(F+1)}$$

$$\boldsymbol{\mu}_I = g_I \mu_N \boldsymbol{I}$$

**Doppler Width**

$$\frac{\Delta\omega_D}{\omega_0} = 2\sqrt{\ln 2}\frac{u}{c} \approx 1.7\frac{u}{c}$$

**Most Probable Speed**

$$u = 2230\sqrt{\frac{T}{300M}} \quad \text{m/s}$$

**Dopplershift**

$$\delta = kv = \frac{\omega v}{c}$$

**Natural Width**

$$\Delta\omega_N = \Gamma = A_{21} = 1\frac{1}{\tau}$$

$$\Delta f_N = \frac{\Delta\omega_N}{2\pi}$$

**Hyper Fine Structure**

$$H = -\boldsymbol{\mu}_I \cdot \boldsymbol{B}_e = A\boldsymbol{I} \cdot \boldsymbol{J}$$

**For S-electrons in Hydrogen-like Systems**

$$A = \frac{2}{3}\mu_0 g_S \mu_B g_I \mu_N \frac{Z^3}{\pi a_0^3 n^3}$$

**Boltzman Distribution**

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-\Delta E/(kT)}$$

**Integrals**

$$\int_0^\infty x^n e^{-\alpha x} \, dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0 \infty x^{2n+1} e^{-\alpha x^2} \, dx = \frac{n!}{2\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2} \, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x^{2n} e^{-\alpha x^2} \, dx = \frac{(2n-1)!!}{2(2\alpha)^n}\sqrt{\frac{\pi}{\alpha}}$$

**Operators**

$$\boldsymbol{p} = -i\hbar\nabla$$

$$\boldsymbol{L} = -i\hbar\boldsymbol{r} \times \nabla$$

$$\boldsymbol{H} = -\frac{\hbar^2}{2m}\nabla^2 + V \qquad \text{(standard)}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

**Dirac Notation**

$$< \boldsymbol{H} >=< \psi|\boldsymbol{H}|\psi >= \int_{\mathbb{R}} \psi^* \boldsymbol{H}\psi \, dv$$

**Commutators**

$$[A,B] = AB - BA$$
$$[A,B] = -[B,A]$$
$$[A,B+C] = [A,B] + [A,C]$$
$$[AB,C] = A[B,C] + [A,C]B$$

**Schrödinger Equation**

$$\boldsymbol{H}\psi = E\psi \qquad \text{(time independent)}$$

$$\boldsymbol{H}\psi = i\hbar\frac{\partial}{\partial t}\psi \qquad \text{(time dependent)}$$

| | |
|---|---|
| Configuration | $\prod n_i l_i^{\omega_i}$ |
| Terms | $L$ and $S(^{2S+1}L)$ |
| Levels | $J$ |
| States (ZE-sublevels) | $M_J$ |
| Hyperfinivåer | $F$ |

| | | | |
|---|---|---|---|
| 1 | $\Delta J = 0, \pm 1$ | $(J = 0 \not\leftrightarrow J' = 0)$ | level |
| 2 | $\Delta M_J = 0, \pm 1$ | $(M_J = 0 \not\leftrightarrow M_{j'} = 0$ if $\Delta J = 0)$ | state |
| 3 | break parity | | configuration |
| 4 | $\Delta l = \pm 1$ | | |
| 5 | $\Delta L = 0, \pm 1$ | $(L = 0 \not\leftrightarrow L' = 0)$ | term |
| 6 | $\Delta S = 0$ | | term |

$1, 2$ are replaced for similar formulas for $F$ and $M_F$ if $F$ is a good quantum number. $5, 6$ only hold if $L$ and $S$ are good quantum numbers.

| | Fine Structure - LS | Hyper Fine Structure - IJ |
|---|---|---|
| interaction | $\beta \boldsymbol{L} \cdot \boldsymbol{S}$ | $A \boldsymbol{I} \cdot \boldsymbol{J}$ |
| moment | $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ | $\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{J}$ |
| eigen-states | $\lvert LSJM_J \rangle$ | $\lvert IJFM_F \rangle$ |
| energy | $\beta/2(J(J+1) - L(L+1) - S(S+1))$ | $A/2(F(F+1) - I(I+1) - J(J+1))$ |
| interval | $E_J - E_{J-1} = \beta J$ | $E_F - E_{F-1} = AF$ |
| | (if $E_{S-O} \ll E_{re}$) | (if $A \gg \Delta E_{quadrupole}$) |

# Waves and Optics

## Oscillations

**Simple harmonic occilations are described by**

$$\frac{d^2 y}{dt^2} + \omega^2 y = 0$$

**With real solutions on the form**

$$y = A \sin(\omega t + \alpha)$$

**Angular Frequency**

$$\omega = \frac{2\pi}{T} = 2\pi f$$

**Energy for Elastic Pendulum**

$$W_{pot} = \frac{ky^2}{2}$$

$$W_{tot} = \frac{m}{2}A^2\omega^2$$

$$\omega = \sqrt{\frac{k}{m}}$$

**Angular Frequency**

$$\omega = \frac{2\pi}{T} = 2\pi f$$

**Wave Number**

$$k = \frac{2\pi}{\lambda}$$

## Wave Equation

**Progressive Plane Wave**

$$s = s_o \sin[2\pi(\frac{t}{T} \pm \frac{x}{\lambda}) + \alpha]$$

**Standing Wave Equation**

$$s = A\cos\left(2\pi\frac{x}{\lambda} + \frac{\phi}{2}\right)\sin\left(2\pi\frac{t}{T} + \frac{\phi}{2}\right)$$

where $\phi$ is the phase shift at origo. Node distance is $\frac{\lambda}{2}$

**The General Wave Equation**

$$\frac{\partial^2 s}{\partial t^2} = v^2\frac{\partial^2 s}{\partial x^2}$$

**Occilation Frequency**

$$f_{\text{occilation}} = |f_1 - f_2|$$

## Sound and Doppler Effect

### Doppler Effect

$$f_m = f_s \frac{v - v_m}{v - v_s}$$

### Supersonic Speed

$$\sin\theta = \frac{v_{sound}}{v_{[planar/[plan]]}} = \frac{1}{M\alpha}$$

### Compressibility coefficient

$$\kappa = -\frac{1}{\Delta P} \cdot \frac{\Delta V}{V}$$

### Sound Pressure

$$p = -\frac{1}{\kappa} \cdot \frac{\partial s}{\partial x}$$

$$p = \mp p_0 \cos\left[2\pi\left(\frac{t}{T} \pm \frac{x}{\lambda}\right)\right]$$

### Pressure Amplitude

$$p_0 = \frac{2\pi s_0}{\kappa\lambda} = Z s_0 \omega$$

### Acoustic Impedance

$$Z = \rho v$$

### Speed of Sound (Fluid and Gas)

$$v = \frac{1}{\sqrt{\kappa\rho}}$$

$$v = \sqrt{\frac{c_p RT}{c_v M}}$$

### Speed of Sound (String and Rod)

$$v = \sqrt{\frac{F}{\mu}}$$

$$v = \sqrt{\frac{E}{\rho}}$$

**Sound Intensity**

$$I = \frac{Z}{2} s_0^2 \omega^2$$

$$I = \frac{p_0^2}{2Z}$$

**Sound Intensity Level**

$$L_I = 10 lg \frac{I}{I_0}$$

$$\text{med } I_0 = 1,0 \cdot 10^{-12} \, W/m^2$$

**Refraction and Transmittance of Sound**

$$R \equiv \frac{I_r ef}{I_i n} = \left( \frac{Z_2 - Z_1}{Z_2 + Z_1} \right)^2$$

$$T \equiv \frac{I_t r}{I_i n} = 1 - R$$

**Harmonics (Strings and Open Cylinders)**

$$f_m = m \cdot f_1 \quad m = 2, 3, 4, ...$$

**Harmonics (Half Open Cylinders)**

$$f_m = (2m - 1) \cdot f_1 \quad m = 2, 3, 4, ...$$

## Light

**DeBroglie wavelength**

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

**Speed of Light**

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$$v = \frac{c}{\sqrt{\mu_r \epsilon_r}}$$

**Intensity EM-Wave**

$$I = \frac{1}{2} \sqrt{\frac{\epsilon_0 \epsilon_r}{\mu_0 \mu_r}} E_0^2 \ \ , \quad B_z = \frac{E_y}{v}$$

**Intensity when two waves are added**

$$I_{tot} = I_1 + I_2 + 2\sqrt{I_1 I_2} < \cos\delta >$$

where $\delta$ is the relative phase between the waves.

**Refractive Index**

$$n \equiv \frac{c}{v} = \sqrt{\mu_r \epsilon_r}$$

**Snell's Law**

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{\lambda_1}{\lambda_2} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

**Boundary Angle for Total Reflection**

$$\alpha_g = \arcsin\left(\frac{n_2}{n_1}\right)$$

**Prism**

$$\sin\left(\frac{A+\delta}{2}\right) = n \cdot \sin\left(\frac{A}{2}\right)$$

Where $A$ is the prisms top angle and $\delta$ the reflection angle.

**Fiber Optics, Numerical Aperture**

$$N.A. \equiv n_0 \sin\theta_m$$

$$N.A. = \sqrt{n_1^2 - n_2^2}$$

## Material Properties for Sound and Light

### Material Properties for Sound and Light

Speed of Sound at 1 atm and 20 °C:

| | |
|---|---|
| Iron | 5950 m/s |
| Glass (Approx) | 5600 m/s |
| Copper | 4760 m/s |
| Lead | 2160 m/s |
| Rubber | 1550 m/s |
| Water | 1461 m/s |
| Mercury | 1407 m/s |
| Methanol | 1143 m/s |
| Ether | 1032 m/s |
| Hydrogen | 1286 m/s |
| Helium | 1008 m/s |
| Air | 343 m/s |
| Oxygen | 326 m/s |
| Carbon dioxide | 269 m/s |

Aucustic Impedance at 1 atm and 20 °C:

| | |
|---|---|
| Hydrogen Gas | 111 $\text{Ns/m}^3$ |
| Air | 412 $\text{Ns/m}^3$ |
| Water | $1,46 \cdot 10^6$ $\text{Ns/m}^3$ |
| Rubber | $1,47 \cdot 10^6$ $\text{Ns/m}^3$ |
| Glycerin | $2,42 \cdot 10^6$ $\text{Ns/m}^3$ |
| Quarts | $13,1 \cdot 10^6$ $\text{Ns/m}^3$ |
| Glass (Approx) | $14 \cdot 10^6$ $\text{Ns/m}^3$ |
| Aluminum | $17,3 \cdot 10^6$ $\text{Ns/m}^3$ |
| Mercury | $19,1 \cdot 10^6$ $\text{Ns/m}^3$ |
| Copper | $33,9 \cdot 10^6$ $\text{Ns/m}^3$ |
| Steel | $46,4$ $\text{Ns/m}^3$ |
| Tungsten | $101 \cdot 10^6$ $\text{Ns/m}^3$ |

Vacuum Wavelengths and Frequencies of Light:

| Color | Wavelength | Frequency |
|---|---|---|
| Violet | $400 - 440$ nm | $749 - 681$ THz |
| Blue | $440 - 480$ nm | $681 - 625$ THz |
| Green | $480 - 560$ nm | $625 - 535$ THz |
| Yellow | $560 - 590$ nm | $535 - 508$ THz |
| Orange | $590 - 620$ nm | $508 - 484$ THz |
| Red | $620 - 700$ nm | $484 - 428$ THz |

## Geometrical Optics

**Refraction in spherical surface**

$$\frac{n_1}{a} + \frac{n_2}{b} = \frac{n_2 - n_1}{R}$$

**Gauss Formula**

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{f}$$

**Lateral Enlargement**

$$M \equiv \frac{y_b}{y_a} \qquad M = -\frac{b}{a}$$

**Focal Length Curved Mirror**

$$f = -\frac{R}{2}$$

**Refractive Power (Lens)**

$$B \equiv \frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

**Lens**

Lens with refractive index $n_1$ in medium with refractive index $n_2$:

$$B \equiv \frac{1}{f} = \left[\frac{n_1}{n_2} - 1\right] \cdot \left[\frac{R_2 - R_1}{R_1 \cdot R_2}\right]$$

**Aparture Number**

$$b_t \equiv \frac{f}{D}$$

**Depth of Field**

$$s \approx \frac{a^2}{1000f} b_t$$

**Angular Magnification of Magnifier**

$$G = \frac{d_0}{f} \text{ where,} \quad d_0 = 25\,\text{cm}$$

**Angular Magnification of Microscope**

$$G = |M_{ob}| \cdot G_{ok} = \frac{L}{f_{ob}} \frac{d_0}{f_{ok}}$$

where the tube length $L = 16\,\text{cm}$

**Angle magnification of the Kepler and Galileo binoculars**

$$G = \left| \frac{f_{ob}}{f_{ok}} \right|$$

**Refraction in a spherical surface**

Positive if: C is to the right of O
Positive if: A is to the left of O
Positive if: B is to the right of O
Positive if: $F_A$ is to the left of O
Positive if: $F_B$ is on the right of O
Image with thin lens in air
Positive if: the lens is convex (gathers light)
Positive if: the object is to the left of the lens
Positive if: the image is to the right of the lens
Positive if: the object is above the optical axis
Positive if: the image is above the optical axis
Positive if: the image is upside up
Image with a curved mirror
Positive if: C is to the right of O (convex)
Positive if: F is to the left of O (concave)
Positive if: A is to the left of O
Positive if: B to the left of O
Positive if: the image is upside up

### Refractive Index for Some Materials

Refractive Index with $\lambda = 589\,\text{nm}$ at 20 °C:

| | |
|---|---|
| Water | $1,333$ |
| Diethyl Ether | $1,353$ |
| Ethanol | $1,361$ |
| Glycerin | $1,455$ |
| Benzene | $1,501$ |
| Carbon Sulfur | $1,628$ |
| Is (0 °C) | $1,31$ |
| NaCl | $1,544$ |
| Polystyrene | $1,59$ |
| Crown Glass (FK5) | $1,487$ |
| Crown Glass (BK7) | $1,517$ |
| Canada balsam | $1,542$ |
| Flint Glass (F2) | $1,620$ |
| Flint Glass (SF10) | $1,728$ |
| Flint Glass (SFS1) | $1,922$ |
| Quarts | $1,458$ |
| Plexiglass | $1,49 - 1,52$ |
| Diamond | $2,417$ |

## Diffraction and Interferance

### Intensity when Diffraction

$$I = I_0 \left( \frac{\sin\beta}{\beta} \right)^2 \qquad \text{with} \qquad \beta = \frac{\pi}{\lambda} b \sin\theta$$

### Diffraction minimun of slit

$$b\sin\theta = m\lambda \qquad \text{where} \qquad m = \pm1, \pm2, \pm3, ...$$

### Diffraction minimum of round opening

$$D\sin\theta = k\lambda$$

$$\text{where} k = 1,22 \quad 2,23 \quad 3,24 \quad 4,25 \quad 5,25...$$

**Rayleigh's Resolution Criterion**

Central top for the first point over the first min for the second point

**Interference if Diffraction is neglected**

$$I = I_0 \left( \frac{\sin N\gamma}{\sin \gamma} \right) \qquad \text{där} \qquad \gamma = \frac{\pi}{\lambda} d \sin \theta$$

**Interference gives main max if**

$$d \sin \theta = m\lambda \qquad \text{där} \qquad m = \pm 1, \pm 2, \pm 3, ...$$

**Visibility**

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$

**Grating, transmission or reflection**

$$d(\sin \alpha_2 + \sin \alpha_1) = m\lambda$$

$$d(\sin \alpha_2 - \sin \alpha_1) = m\lambda$$

**Max or min in case of interference in thin layers**

$$2n_2 d \cos \alpha_2 = m\lambda \qquad \text{där} \qquad m = 0, \pm 1, \pm 2, ...$$

**Finesse in Fabry-Perot interferometer**

$$F = \frac{\Delta f}{\delta f} \qquad \text{where} \qquad \Delta f = \frac{c}{2d}$$

**Airy Function**

$$T = \frac{1}{1 + \left[ \frac{4r^2}{(1-r^2)^2} \right] \sin^2 \left( \frac{\delta}{2} \right)}$$

## Fresnel Diffraction

### Fresnel-Kirchhoff

$$E_p = \frac{-ik}{2\pi} E_s e^{-i\omega t} \iint_{Obstacle} F(\theta) \frac{e^{ik(r+r')}}{rr'} \, dA$$

**Skewness Factor**

$$F(\theta) = \frac{1 + \cos \theta}{2}$$

**Raius of Fresnel Zones**

$$R_n \approx \sqrt{nL\lambda} \quad \text{where} \quad \frac{1}{L} = \frac{1}{p} + \frac{1}{q}$$

## Polarization

**Malus Law**

$$I = I_0 \cos^2 \theta$$

**Phase difference in birefringent material**

$$\phi = \frac{2\pi}{\lambda} d|n_e - n_o|$$

**Reflection at normal incidence**

$$R \equiv \frac{I_{ref}}{I_{in}} = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2$$

**Brewster's Angle in Air**

$$\theta_{air} = \arctan n$$

**Wiens Displacement Law**

$$\lambda_{max} \cdot T = 2,898 \cdot 10^3 \, \mu m \cdot K$$

# Thermodynamics

## Thermodynamics

**Heat Expansion**

$$\frac{\Delta L}{L} = \alpha \Delta T, \quad \frac{\Delta V}{V} = \beta \Delta T, \quad \beta = 3\alpha$$

**Heat**

$$Q = mc\Delta T, \quad l_s = \frac{Q_s}{m}, \quad l_{\text{å}} = \frac{Q_{\text{å}}}{m}$$

**Fluid Pressure**

$$p_{tot} = p_{\text{fluid}} + p_{\text{air}} = \rho g h + p_{\text{air}}$$

**Ideal Gas Law**

$$pV = NkT \quad \text{or} \quad pV = nRT$$

$$\text{where} \quad n = \frac{m_{tot}}{M} = \frac{N}{N_A} \quad \text{and} \quad R = kN_A$$

**Gas Density and Particle Density**

$$\rho = \frac{m_{tot}}{V} = \frac{pM}{RT}, \quad n_o = \frac{N}{V} = \frac{p}{kT}$$

**Barometric Height Formula**

$$p = p_0 e^{-\rho_0 g h / p_0}, \quad h = \frac{p_0}{\rho_0 g} \ln \frac{p_0}{p}$$

**Relative Moisture**

$$R_M = \frac{p_{\text{water}}}{p_{\text{saturation}}}$$

**Van der Waal's Equation**

$$\left(p + a\frac{n^2}{V^2}\right)(V - nb) = nRT$$

**Critical Point**

$$V_k = 3nb, \quad T_k = \frac{8a}{27Rb}, \quad p_k = \frac{a}{27b^2}$$

**Molecule Radius**

$$r = \left(\frac{3b}{16\pi N_A}\right)^{1/3}$$

**Vapor Pressure Curve**

$$p = Ae^{-Ml_{\text{å}}/(RT)}$$

**Reynolds Number**

$$Re = \frac{\rho v d}{\eta}, \quad Re < 2300 \text{ laminar}$$

**Volume Flow**

$$\Phi = \frac{\mathrm{d}V}{\mathrm{d}t} = A_1 v_1 = A_2 v_2$$

**Bernoullis Equation**

$$p_1 + \frac{\rho v_1^2}{2} + \rho g y_1 = p_2 + \frac{\rho v_2^2}{2} + \rho g y_2$$

**Poiseuilles Law**

$$\Phi = \frac{\pi R^4}{8\eta} \frac{(p_1 - p_2)}{L}$$

**Pressure (Microscopic)**

$$p = \frac{2}{3} n_o \frac{m_{\text{en}}}{2} \langle v^2 \rangle = \frac{2}{3} n_o \left\langle W_{\text{kin}} \right\rangle_{\text{en}}$$

**Temperature (Microscopic)**

$$\left\langle W_{\text{kin}} \right\rangle_{\text{en}} = \frac{3}{2} kT$$

**Inner Energy (change)**

$$\Delta U = \frac{f}{2} N k \Delta T = \frac{f}{2} n R \Delta T$$

**First Theorem**

$$Q = \Delta U + W \quad \text{with} \quad W = \int_1^2 p dV$$

**Isokor**

$$W \equiv 0$$

**Isobar**

$$W = p\,(V_2 - V_1)$$

**Isotherm**

$$W = nRT \ln\left(\frac{V_2}{V_1}\right)$$

**Adiabat**

$$W = -\Delta U$$

**Molar Heat Capacity**

$$c_V = \frac{f}{2} R, \quad c_p = c_V + R$$

**Adiabat(Poissons Equations)**

$$T_1 V_1^{(\gamma-1)} = T_2 V_2^{(\gamma-1)}$$
$$p_1 V_1^{\gamma} = p_2 V_2^{\gamma}$$

**Quotient**

$$\gamma \equiv \frac{C_p}{C_V} = \frac{c_p}{c_V} = 1 + \frac{2}{f}$$

**Circuit Process**

$$Q_{\text{net}} = W_{\text{net}} = \oint p dV$$

**Efficiency**

$$\eta = \frac{W_{\text{net}}}{Q_{\text{in}}} = \frac{Q_{\text{in}} - |Q_{\text{out}}|}{Q_{\text{in}}} = 1 - \frac{|Q_{\text{out}}|}{Q_{\text{in}}}$$

**Ideal Efficiency**

$$\eta = \frac{T_{\text{warm}} - T_{\text{cold}}}{T_{\text{warm}}} = 1 - \frac{T_{\text{cold}}}{T_{\text{warm}}}$$

**Cold Factor (def. and Ideal)**

$$K_f \equiv \frac{Q_{\text{in}}}{|W_{\text{net}}|}, \quad K_f = \frac{T_{\text{cold}}}{T_{\text{warm}} - T_{\text{cold}}}$$

**Heat Factor (def. and Ideal)**

$$V_f \equiv \frac{Q_{\text{out}}}{|W_{\text{net}}|}, \quad V_f = \frac{T_{\text{warm}}}{T_{\text{warm}} - T_{\text{cold}}}$$

**Gauss Distribution**

$$f(v_z) = \sqrt{\frac{m_{\text{en}}}{2\pi kT}} e^{-m_{\text{en}} v_z^2/(2kT)}$$

**Maxwell–Boltzmann Distribution**

$$f(v) = 4\pi v^2 \left(\frac{m_{\text{en}}}{2\pi kT}\right)^{3/2} e^{-m_{\text{en}} v^2/(2kT)}$$

**Averages**

$$\langle v \rangle = \sqrt{\frac{8kT}{\pi m_{\text{en}}}}, \quad \langle v \rangle = 2\langle |v_x| \rangle$$
$$\langle W_{\text{kin}} \rangle = \left\langle \frac{m_{\text{en}} v^2}{2} \right\rangle = \frac{m_{\text{en}}}{2} \langle v^2 \rangle = \frac{3}{2} kT$$

**Collision Number (per second and square meter)**

$$n^* = \frac{n_o}{4}\langle v\rangle$$

**Mean Free Path**

$$l = \frac{1}{n_o \pi d^2 \sqrt{2}}$$

**Heat Conduction (General and Rod)**

$$P = -\lambda A \frac{\mathrm{d}T}{\mathrm{d}x}, \quad P = \lambda A \frac{T_1 - T_2}{L}$$

**Heat Transfer**

$$P = \alpha A \Delta T$$

**Radiation**

$$P_{\text{ideal}} = \sigma A T^4, \quad P_{\text{real}} = e P_{\text{ideal}}$$

## Tables

**Saturation Pressure for Water**

| $t/°C$ | Water/kPa |
|---|---|
| $-30$ | 0.0381 |
| $-20$ | 0.103 |
| $-15$ | 0.165 |
| $-10$ | 0.260 |
| $-5$ | 0.401 |
| 0 | 0.610 |
| 5 | 0.872 |
| 10 | 1.23 |
| 15 | 1.70 |
| 20 | 2.34 |
| 25 | 3.17 |
| 30 | 4.24 |
| 35 | 5.64 |
| 40 | 7.37 |
| 50 | 12.3 |
| 60 | 19.9 |
| 70 | 31.2 |
| 80 | 47.3 |
| 90 | 70.1 |
| 100 | 101.3 |
| 110 | 143.2 |
| 120 | 198.4 |
| 130 | 270.0 |

**Length expansion coefficient at 20 °C and normal air pressure.**

| Substance | $\alpha/(10^{-}6\mathrm{K}^{-}1)$ | Substance | $\alpha/(10^{-}6\mathrm{K}^{-}1)$ |
|---|---|---|---|
| Aluminum | 23 | Glass (typical) | 6.0 |
| Silver | 19 | Tungsten | 4.3 |
| Brass (Cu + Zn) | 19 | Marble (typical) | 2.5 |
| Copper | 17 | Invar (Fe + Ni) | 2.0 |
| Iron | 12 | Graphite | 2.0 |
| Steel | 11 | Diamond | 1.2 |
| Platinum | 9.0 | Quartz | 0.4 |

# Constants

## Constants

### Constants

| Name | Variable | Value | Unit |
|---|---|---|---|
| Speed of light in a vacuum | $c$ | $299\,792\,458$ | m/s |
| Planck's constant | $h$ | $6.626\,070\,15 \cdot 10^{-34}$ | Js |
| Planck's constant | $h$ | $4.135\,667\,87 \cdot 10^{-15}$ | eVs |
| Planck's constant | $\hbar$ | $1.054\,573 \cdot 10^{-34}$ | Js |
| Planck's constant | $\hbar$ | $0.658\,212 \cdot 10^{-15}$ | eVs |
| The Elemental Charge | $e$ | $1.602\,176\,634 \cdot 10^{-19}$ | C |
| Bohr Radius | $a_0$ | $5.291\,772\,109\,03 \cdot 10^{-11}$ | m |
| Electron Mass | $m_e$ | $9.109\,383\,7015 \cdot 10^{-31}$ | kg |
| Electron Mass | $m_e$ | $0.510\,998\,954$ | MeV/c$^2$ |
| Proton Mass | $m_p$ | $1.672\,621\,923\,69 \cdot 10^{-27}$ | kg |
| Proton Mass | $m_p$ | $938.272\,096$ | MeV/c$^2$ |
| Proton Mass | $m_p$ | $1836.152\,673\,43$ | $m_e$ |
| Neutron Mass | $m_n$ | $1.674\,927\,498\,04 \cdot 10^{-27}$ | kg |
| Neutron Mass | $m_n$ | $939.565\,428$ | MeV/c$^2$ |
| Neutron Mass | $m_n$ | $1838.683\,661\,73$ | $m_e$ |
| Boltzmanns Constant | $k$ | $1.380\,649 \cdot 10^{-23}$ | J/K |
| Boltzmanns Constant | $k$ | $8.617\,333\,6333 \cdot 10^{-5}$ | eV/K |
| Avogadros Constant | $N_A$ | $6.022\,140\,76 \cdot 10^{23}$ | mol$^{-1}$ |
| Rydbergs Constant | $R_y$ | $\frac{\hbar^2}{2ma_0^2}$ | |
| Rydbergs Constant | $R_y$ | $13.6057$ | eV |
| Rydbergs Constant | $R_y$ | $109\,737.32$ | cm$^{-1}$ |
| The General Gas Constant | $R$ | $8.314\,462\,618$ | J/(mol $\cdot$ K) |
| The Fine Structure Constant | $\alpha$ | $\frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{1}{137.036}$ | |
| Dielectric Constant for Vacuum | $\varepsilon_0$ | $0.885\,419 \cdot 10^{-11}$ | As/Vm |
| Permeability of Vacuum | $\mu_0$ | $1.256\,637\,062\,12 \cdot 10^{-6}$ | Vs/Am |
| Permeability of Vacuum | $\mu_0$ | $4\pi \cdot 10^{-7}$ | Vs/Am |
| The Bohr Magneton | $\mu_B$ | $\frac{e\hbar}{2m} = 9.274\,010\,0783 \cdot 10^{-24}$ | Am$^2$ |

# Prefix

## Prefix

### SI-prefix

| SI-prefix | Symbol | Decimal |
|---|---|---|
| Yotta | Y | $1e24$ |
| Zetta | Z | $1e21$ |
| Exa | E | $1e18$ |
| Peta | P | $1e15$ |
| Tera | T | $1e12$ |
| Giga | G | $1e9$ |
| Mega | M | $1e6$ |
| Kilo | k | $1e3$ |
| Hecto | h | $1e2$ |
| Deca | da | $1e1$ |
| Deci | d | $1e-1$ |
| Centi | c | $1e-2$ |
| Milli | m | $1e-3$ |
| Micro | µ | $1e-6$ |
| Nano | n | $1e-9$ |
| Pico | p | $1e-12$ |
| Femto | f | $1e-15$ |
| Atto | a | $1e-18$ |
| Zepto | z | $1e-21$ |
| Yocto | y | $1e-24$ |

# Periodic Table

# Unit Conversion