Integrals and Identities

Some integrals

Indefinite Integrals

$$\int \frac{x}{ax+b}dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b|$$

$$\int \frac{1}{x(ax+b)}dx = -\frac{1}{b}\ln\left|\frac{ax+b}{x}\right|$$

$$\int \frac{ax+b}{fx+g}dx = \frac{ax}{f} + \frac{bf-ag}{f^2}\ln|fx+g|$$

$$\int \frac{x}{(ax+b)(fx+g)}dx = \frac{1}{bf-ag}\left[\frac{b}{a}\ln|ax+b| - \frac{g}{f}\ln|fx+g|\right]$$

Definition: $\chi = \begin{cases} \frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}} & \text{if } 4ac > b^2 \\ \frac{1}{a(p-q)}\ln\left|\frac{x-p}{x-q}\right| & \text{if } 4ac < b^2 \end{cases}$

Where p and q are the roots of $ax^2 + bx + c = 0$.

$$\int \frac{1}{ax^2+bx+c}dx = \chi$$

$$\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{2a}\chi$$

$$\int \frac{x^2}{ax^2+bx+c}dx = \frac{x}{a} - \frac{b}{2a^2}\ln|ax^2+bx+c| + \frac{b^2-2ac}{2a^2}\chi$$

$$\int \frac{1}{(ax^2+bx+c)^2}dx = \frac{2ax+b}{(4ac-b^2)(ax^2+bx+c)} + \frac{2a}{(4ac-b^2)}\chi$$

$$\int \frac{x}{(ax^2+bx+c)^2}dx = -\frac{bx+2c}{(4ac-b^2)(ax^2+bx+c)} - \frac{b}{(4ac-b^2)}\chi$$

$$\int \sqrt{ax+b}\,dx = \frac{2}{3a}(ax+b)^{3/2}$$

$$\int x\sqrt{ax+b}\,dx = \frac{2(3ax-2b)}{15a^2}(ax+b)^{3/2}$$

$$\int \frac{1}{\sqrt{ax+b}}dx = \frac{2\sqrt{ax+b}}{a}$$

$$\int \frac{x}{\sqrt{ax+b}}dx = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b}$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}$$

$$\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}(a^2-x^2)^{3/2}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a}$$

$$\int \frac{x}{\sqrt{a^2-x^2}}dx = -\sqrt{a^2-x^2}$$

$$\int \frac{1}{x\sqrt{a^2-x^2}}dx = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2-a^2}\right|$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left|x + \sqrt{x^2-a^2}\right|$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2+a^2}\right|$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left|x + \sqrt{x^2+a^2}\right|$$

$$\int \frac{1}{\sin ax}dx = \frac{1}{a}\ln\left|\tan\frac{ax}{2}\right|$$

$$\int \frac{1}{\cos ax}dx = \frac{1}{a}\ln\left|\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right|$$

$$\int \tan ax\, dx = -\frac{1}{a}\ln|\cos ax|$$

$$\int \cot ax \, dx = \frac{1}{a} \ln |\sin ax|$$

$$\int x \sin x \, dx = \sin x - x \cos x$$

$$\int x \sin^2 x \, dx = \frac{x^2}{4} - \frac{x \sin 2x}{4} - \frac{\cos 2x}{8}$$

$$\int \sin^2 x \, dx = \frac{1}{2}(x - \sin x \cos x)$$

$$\int \ln x \, dx = x \ln |x| - x$$

$$\int x e^{ax} \, dx = \frac{e^{ax}}{a^2}(ax - 1)$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)$$

Definite Integrals

$$\int_0^\infty x^{2n} \cdot e^{-ax^2} \, dx = \frac{(2n-1)!!}{2(2a)^n} \sqrt{\frac{\pi}{a}}$$

Where $a > 0$, $n! =$ negative integer.

$$\int_0^\infty x^{2n+1} \cdot e^{-ax^2} \, dx = \frac{n!}{2a^{n+1}}$$

$$\int_0^\infty x^k \cdot e^{-ax} \, dx = \Gamma(k+1) \cdot a^{-(k+1)}$$

$$I_k = \int_0^\infty \frac{x^k}{e^x - 1} \, dx = \Gamma(k+1) \cdot \zeta(k+1)$$

$$J_k = \int_0^\infty \frac{x^k}{e^x + 1} \, dx = (1 - 2^{-k}) \cdot I_k$$

$$\zeta(k) = \sum_{n=1}^\infty \frac{1}{n^k}$$

$$\int_0^{\pi/2} \sin^{2a+1} x \cdot \cos^{2b+1} x \, dx = \frac{\Gamma(a+1) \cdot \Gamma(b+1)}{2\Gamma(a+b+2)}$$

Stirling's approximation

$$N! \approx \sqrt{2\pi N} \left(\frac{N}{e}\right)^N \qquad N \gg 1$$

Error Function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} \, d\xi$$

$$\text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\xi^2} \, d\xi$$

$$\text{erf}(\infty) = 1$$

Power Series

Power Series

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

$$\sin(x) = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots$$

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{1}{15!}x^5 + \dots |x| < \frac{\pi}{2}$$

$$\ln(1 + x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots |x| < 1$$

$$(1 + x)^a = 1 + \frac{a}{1!}x + \frac{a(a-1)}{2!}x^2 + \dots$$

$$\sqrt{1 + x} = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$$

$$\frac{1}{\sqrt{1 + x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots$$

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots |x| < 1$$

$$\arcsin(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \dots |x| < 1$$

$$\cosh(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots$$

$$\sinh(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots$$

Trigonometric Functions

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$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$$

$$\sin(2\alpha) = 2 \sin \alpha \cos \alpha$$

$$\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha = 2\cos^2 \alpha - 1 = 1 - 2\sin^2 \alpha$$

$$\sin(3\alpha) = 3 \sin \alpha - 4 \sin^3 \alpha$$

$$\cos(3\alpha) = 4 \cos^3 \alpha - 3 \cos \alpha$$

$$\sin^2 \frac{\alpha}{2} = \frac{1}{2}(1 - \cos \alpha)$$

$$\cos^2 \frac{\alpha}{2} = \frac{1}{2}(1 + \cos \alpha)$$

$$\sin \alpha + \cos \beta = 2 \sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta)$$

$$\sin \alpha - \cos \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta)$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta)$$

$$\cos \alpha - \cos \beta = -2 \sin \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta)$$

$$\sin \alpha = \frac{1}{2i}(e^{i\alpha} - e^{-i\alpha})$$

$$\cos \alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha})$$

Hyperbolic Functions

Hyperbolic Functions

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$