Vector Analysis

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Vector Products

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

where $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ are unit vectors.

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$$

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) + \boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a}) + \boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b}) = 0$$

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})$$

$$(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d}) = \boldsymbol{c}((\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{d}) - \boldsymbol{d}((\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c})$$

Gradient, Divergence, Curl and The Laplace Operator

$$\mathrm{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\right)$$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$$\begin{aligned} \mathrm{div}\boldsymbol{a} = \nabla\cdot\boldsymbol{a} &= \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 a_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta a_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial a_\varphi}{\partial\varphi} \end{aligned}$$

$$\begin{aligned} \mathrm{rot}\boldsymbol{a} = \nabla\times\boldsymbol{a} &= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}, \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}, \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) \\ &= \left(\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta a_\varphi) - \frac{\partial a_\theta}{\partial\varphi}\right), \frac{1}{r\sin\theta}\frac{\partial a_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(r a_\varphi), \frac{1}{r}\frac{\partial}{\partial r}(r a_\theta) - \frac{1}{r}\frac{\partial a_r}{\partial\theta}\right) \end{aligned}$$

$$\begin{aligned} \Delta &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \end{aligned}$$

$$\Delta f(r) = \frac{1}{r}\frac{d^2}{dr^2}(rf), \quad r \neq 0$$

$$\begin{gathered} \nabla\times(\nabla\mathcal{U}) = 0 \\ \nabla\cdot(\nabla\mathcal{U}) = \nabla\mathcal{U} \\ \nabla\cdot(\nabla\times\boldsymbol{A}) = 0 \\ \nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A}) - \Delta\boldsymbol{A} \\ \nabla\cdot(\mathcal{U}V) = \mathcal{U}\Delta V + 2\nabla\mathcal{U}\cdot\nabla V + V\Delta\mathcal{U} \\ \nabla\cdot(\mathcal{U}\boldsymbol{V}) = \mathcal{U}\Delta\boldsymbol{V} + 2(\nabla\mathcal{U}\cdot\nabla)\boldsymbol{V} + \boldsymbol{V}\Delta\mathcal{U} \\ \nabla\times(\mathcal{U}\boldsymbol{V}) = \mathcal{U}\nabla\times\boldsymbol{V} + (\nabla\mathcal{U})\times\boldsymbol{V} \\ \nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B}) \\ \nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} + (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} \\ \nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla\cdot\boldsymbol{B}) - \boldsymbol{B}(\nabla\cdot\boldsymbol{A}) \end{gathered}$$

Gauss' theorem

$$\oint_{S(V)} \boldsymbol{a}\cdot d\boldsymbol{S} = \int_V (\nabla\cdot\boldsymbol{a})dV$$

Where dV in polar coordinates are $r^2\sin\theta\ dr\ d\theta\ d\varphi$

Stoke's theorem

$$\oint_{C(S)} \boldsymbol{a}\cdot d\boldsymbol{l} = \int_S (\nabla\times\boldsymbol{a})\cdot d\boldsymbol{S}$$

Where S is an arbitrary surface with border $C(S)$

Green's theorem

$$\oint_{S(V)} (\Psi\nabla\varphi - \varphi\nabla\Psi)\cdot d\boldsymbol{S} = \int_V (\Psi\Delta\varphi - \varphi\Delta\Psi)dV$$